

# ENTIRE SPACELIKE RADIAL GRAPHS IN THE MINKOWSKI SPACE, ASYMPTOTIC TO THE LIGHT-CONE, WITH PRESCRIBED SCALAR CURVATURE

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**ABSTRACT.** Existence and uniqueness in  $\mathbb{R}^{n,1}$  of entire spacelike hypersurfaces contained in the future of the origin  $O$  and asymptotic to the light-cone, with scalar curvature prescribed at their generic point  $M$  as a negative function of the unit vector  $\overrightarrow{Om}$  pointing in the direction of  $\overrightarrow{OM}$ , divided by the square of the norm of  $\overrightarrow{OM}$  (a dilation invariant problem). The solutions are seeked as graphs over the future unit-hyperboloid emanating from  $O$  (the hyperbolic space); radial upper and lower solutions are constructed which, relying on a previous result in the Cartesian setting, imply their existence.

**RESUME.** Existence et unicité dans  $\mathbb{R}^{n,1}$  d'hypersurfaces entières de genre espace contenues dans le futur de l'origine  $O$  et asymptotes au cône de lumière, dont la courbure scalaire est prescrite au point générique  $M$  comme fonction négative du vecteur unité  $\overrightarrow{Om}$  pointant en direction de  $\overrightarrow{OM}$ , divisée par le carré de la norme du vecteur  $\overrightarrow{OM}$  (un problème invariant par homothétie). Les solutions sont cherchées comme graphes sur l'hyperboloïde-unité futur émanant de  $O$  (l'espace hyperbolique); des solutions supérieure et inférieure radiales sont construites qui, d'après un résultat antérieur en cartésien, impliquent l'existence de telles solutions.

## INTRODUCTION

The Minkowski space  $\mathbb{R}^{n,1}$  is the affine Lorentzian manifold  $\mathbb{R}^n \times \mathbb{R}$  endowed with the metric

$$ds^2 = dX'^2 - dX_{n+1}^2, \text{ where } dX'^2 = dX_1^2 + \dots + dX_n^2,$$

setting  $X = (X', X_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ , and time-oriented by  $dX_{n+1} > 0$ . Distinguishing the origin  $O$  of  $\mathbb{R}^{n,1}$ , let

$$\mathbb{H} = \{x \in \mathbb{R}^{n,1} \mid |\overrightarrow{Ox}|^2 = |x'|^2 - x_{n+1}^2 = -1, x_{n+1} > 0\},$$

be the future unit-hyperboloid, model of the hyperbolic space in  $\mathbb{R}^{n,1}$ . If  $\varphi$  is a real function defined on  $\mathbb{H}$ , we define the *radial graph* of  $\varphi$  by

$$\text{graph}_{\mathbb{H}} \varphi = \{X \in \mathbb{R}^{n,1}, \overrightarrow{OX} = e^{\varphi(x)} \overrightarrow{Ox}, x \in \mathbb{H}\}.$$

This is a hypersurface contained in the future open solid cone

$$C^+ = \{X \in \mathbb{R}^{n,1} \mid X_{n+1} > |X'|\}.$$

We say that  $\varphi$  is spacelike if its graph is a spacelike hypersurface, which means that the metric induced on it is Riemannian. Conversely, a spacelike and connected hypersurface in  $C^+$  is the radial graph of a uniquely determined function  $\varphi : \mathbb{H} \rightarrow \mathbb{R}$ .

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The first author was supported by the project UNAM-PAPITT IN 101507; the second author is supported by the CNRS.

**MSC 2000:** 53C40, 35J65, 34C11.

Of course, such a graph may also be considered as the Cartesian graph of some function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$

$$\text{graph}_{\mathbb{R}^n} u = \{(x', u(x')), x' \in \mathbb{R}^n\},$$

and the correspondence between the two representations is bijective passing from the Cartesian chart  $X = (X', X_{n+1})$  restricted to  $C^+$ , to the polar chart  $(x, \rho) \in \mathbb{H} \times (0, \infty)$  of  $C^+$  defined by:

$$\rho = \sqrt{-|\overrightarrow{OX}|^2}, \quad \overrightarrow{Ox} = \frac{1}{\rho} \overrightarrow{OX}.$$

Recall that the principal curvatures  $(\kappa_1, \dots, \kappa_n)$  at a point of a spacelike hypersurface are the eigenvalues of its curvature endomorphism  $dN$ , where  $N$  is the future oriented unit normal field, and the  $m^{\text{th}}$  mean curvature (denoted by  $H_m$ ) is the  $m^{\text{th}}$  elementary symmetric function of its principal curvatures:  $H_m = \sigma_m(\kappa_1, \dots, \kappa_n)$ . For each real  $\lambda > 0$ , the cone  $C^+$  is globally invariant under the ambient dilation  $X \mapsto \lambda X$  of  $\mathbb{R}^{n,1}$  and the above  $m$ -th mean curvature is  $(-m)$ -homogeneous; specifically, it transforms like  $H_m(\lambda X) = \lambda^{-m} H_m(X)$ . It is thus natural to pose, as in [6, Theorem 1], the following inverse problem for  $H_m$ : given a positive function  $h > 0$  on  $\mathbb{H}$  tending to 1 at infinity, find a spacelike hypersurface  $\Sigma$  in  $C^+$ , asymptotic to  $\partial C^+$  at infinity, such that, for each point  $X \in \Sigma$ , the  $m$ -th mean curvature of  $\Sigma$  at  $X$  is given by:

$$(1) \quad \frac{1}{\binom{n}{m}} H_m(X) = \frac{1}{(-|\overrightarrow{OX}|^2)^{\frac{m}{2}}} [h(x)]^m, \text{ with } \overrightarrow{Ox} = \frac{\overrightarrow{OX}}{\sqrt{-|\overrightarrow{OX}|^2}}.$$

By construction, this problem is dilation invariant; moreover, as explained below, the positivity of  $h$  makes it elliptic. Actually, introducing the positivity cone [9] of  $\sigma_m$ :

$$\Gamma_m = \{\kappa \in \mathbb{R}^n, \forall i = 1, \dots, m, \sigma_i(\kappa) > 0\},$$

and recalling McLaurin's inequalities (satisfied on  $\Gamma_m$ ):

$$0 < (H_m)^{\frac{1}{m}} \leq (H_{m-1})^{\frac{1}{m-1}} \leq \dots \leq H_2^{\frac{1}{2}} \leq H_1,$$

we note that, if a hypersurface  $\Sigma = \text{graph}_{\mathbb{R}^n} u$  solves (1) with the asymptotic condition, then the time-function  $u$  must assume a minimum on  $\Sigma$  and, as readily checked (using *e.g.* [3, p.245]), the principal curvatures of  $\Sigma$  at such a minimum point of  $u$  must lie in  $\Gamma_m$ . Now equation (1) combined with McLaurin's inequalities forces the principal curvatures of  $\Sigma$  to stay in  $\Gamma_m$  *everywhere*. Let us call any spacelike hypersurface of  $C^+$  having this property,  $m$ -admissible; accordingly, a function  $\varphi : \mathbb{H} \rightarrow \mathbb{R}$  (resp.  $u : \mathbb{R}^n \rightarrow \mathbb{R}$ ) is called  $m$ -admissible, provided  $\text{graph}_{\mathbb{H}} \varphi$  (resp.  $\text{graph}_{\mathbb{R}^n} u$ ) is so. The condition of  $m$ -admissibility is local (and open); one may thus speak of a function  $\varphi : \mathbb{H} \rightarrow \mathbb{R}$  being  $m$ -admissible *at a point* (hence nearby) whenever  $\text{graph}_{\mathbb{H}} \varphi$  is so at that point. We will seek the solution hypersurface  $\Sigma$  as the radial graph of some  $m$ -admissible function  $\varphi : \mathbb{H} \rightarrow \mathbb{R}$  vanishing at infinity (to comply with the asymptotic condition). Equation (1) then reads

$$(2) \quad F_m(\varphi) = h,$$

with the radial operator  $F_m$  defined by:

$$F_m(\varphi) = e^\varphi \left[ \frac{1}{\binom{n}{m}} H_m(X) \right]^{\frac{1}{m}}, \quad X \in \text{graph}_{\mathbb{H}} \varphi.$$

For brevity, we will not compute here explicitly the general expression of the operator  $F_m$  (keeping it for a further study) – its restriction to radial functions will

suffice (see section 3.3 below). We will rely instead on the well-known corresponding Cartesian expression (see *e.g.* [2]) combined with a few basic properties of  $F_m$  recorded in the next section (and proved with elementary arguments).

Furthermore, we will essentially restrict to the case  $m = 2$  (and freely say 'admissible', for short, instead of '2-admissible'). Since  $H_2$  is related to the scalar curvature  $S$  by  $S = -2H_2$ , our present study is really about the prescription of the scalar curvature, at a generic point  $X$  of a radial graph, as a negative function of  $x \in \mathbb{H}$  (with  $x$  given as in (1)) divided by the square of the norm of  $\overrightarrow{OX}$ . Aside from the origin  $O$  of the ambient space  $\mathbb{R}^{n,1}$ , we will distinguish a point  $o$  in  $\mathbb{H}$  and set  $r = r(x)$  for the hyperbolic distance from  $o$  to  $x \in \mathbb{H}$ ; accordingly, a function on  $\mathbb{H}$  will be called *radial* whenever it factors through a function of  $r$  only. Our main result is the following:

**Theorem 1.** *For  $\alpha \in (0, 1)$ , let  $h : \mathbb{H} \rightarrow (0, \infty)$  be a function of class  $C^{2,\alpha}$  with*

$$\lim_{r(x) \rightarrow +\infty} h(x) = 1 .$$

*Assume that the functions  $h^-$  and  $h^+$  defined on  $\mathbb{R}^+$  by*

$$h^-(r) = \sup_{r(x)=r} h(x) \text{ and } h^+(r) = \inf_{r(x)=r} h(x)$$

*satisfy*

$$\int_0^{+\infty} (h^- - 1)_+ dr < +\infty , \quad \int_0^{+\infty} (1 - h^+)_+ dr < +\infty ,$$

*where  $(h^- - 1)_+$  (resp.  $(1 - h^+)_+$ ) means the positive part of  $h^- - 1$  (resp.  $1 - h^+$ ). Then the equation*

$$F_2(\varphi) = h$$

*has a unique admissible solution of class  $C^{4,\alpha}$  such that  $\lim_{r(x) \rightarrow +\infty} \varphi(x) = 0$ .*

**Remark 1.** From Lemma 4 below, anytime the function  $h$  is radial, the integral convergence conditions of Theorem 1 appears necessary for the existence of bounded solutions.

An analogous problem in the Euclidean setting is solved for the Gauss curvature in [6, Théorème 1], and in [12, 5] some related problems are studied. In the Lorentzian setting, the prescription of the mean curvature for entire graphs is studied in [1] and that of the Gauss curvature in [11, 8, 4]. In [3], the scalar curvature is prescribed in Cartesian coordinates  $x_{n+1} = u(x_1, \dots, x_n)$ .

The outline of the paper is as follows. In section 1, we prove that there exists at most one solution vanishing at infinity for equation (2) with  $m \in \{1, \dots, n\}$ . In section 2, relying on [3], we prove the existence of a solution when  $m = 2$ , provided upper and lower barriers are known. The latter are constructed, as radial functions, in section 3.

## 1. UNIQUENESS

We first require a few basic properties of the operator  $F_m$ . It is a nonlinear second order scalar differential operator defined on  $m$ -admissible real functions on  $\mathbb{H}$ . The dilation invariance of (1) implies the identity:

$$(3) \quad F_m(\psi + c) \equiv F_m(\psi) ,$$

for every  $m$ -admissible function  $\psi : \mathbb{H} \rightarrow \mathbb{R}$  and constant  $c$ ; linearizing at  $\psi$  yields

$$dF_m(\psi)(1) \equiv 0 .$$

Furthermore, we have:

**Lemma 1.** *For each  $m$ -admissible function  $\psi$ , the linear differential operator  $dF_m(\psi)$  is elliptic everywhere on  $\mathbb{H}$ , with positive-definite symbol.*

Summarizing for later use, the expression of  $dF_m(\psi)$ , in the chart  $x' \in \mathbb{R}^n$  of  $\mathbb{H}$ , at a fixed  $m$ -admissible function  $\psi$  reads like:

$$(4) \quad \delta\psi \mapsto dF_m(\psi)(\delta\psi) = \sum_{1 \leq i, j \leq n} B_{ij} \frac{\partial^2}{\partial x'_i \partial x'_j} (\delta\psi) + \sum_{i=1}^n B_i \frac{\partial}{\partial x'_i} (\delta\psi),$$

with the  $n \times n$  matrix  $(B_{ij})$  symmetric positive definite (depending on  $\psi$ , of course, like the  $B_i$ 's). We now proceed to proving Lemma 1.

*Proof :* We require the Cartesian operator  $v \mapsto G_m(v) := F_m(\psi)$  defined on  $m$ -admissible functions  $v : \mathbb{R}^n \rightarrow \mathbb{R}$  by:

$$(5) \quad \text{graph}_{\mathbb{R}^n} v = \text{graph}_{\mathbb{H}} \psi.$$

The ellipticity of  $dG_m(v)$  and the positive-definiteness of its symbol are well-known [10, 13, 2]. Its expression thus starts out like

$$dG_m(v)(\delta v) = \sum_{1 \leq i, j \leq n} A_{ij} \frac{\partial^2}{\partial X'_i \partial X'_j} (\delta v) + \text{lower order terms},$$

with the matrix  $(A_{ij})$  symmetric positive definite. The  $m$ -admissible function  $\psi$  on  $\mathbb{H}$  such that (5) holds, is related to  $v$ , in the chart  $x' = (x_1, \dots, x_n) \in \mathbb{R}^n$ , by:

$$v(X') = \sqrt{1 + |x'|^2} \exp[\psi(x')], \text{ with } \overrightarrow{OX'} = e^{\psi(x')} \overrightarrow{Ox'}.$$

Varying  $\psi$  by  $\delta\psi$  thus yields for the corresponding variation  $\delta v$  of  $v$  the following expression:  $\delta v(X') = w(X') \delta\psi(x')$ , with  $w(X') = \left[ v - \sum_{i=1}^n X'_i \frac{\partial v}{\partial X'_i} \right] (X')$ . Since the graph lies in  $C^+$  and it is spacelike, we have  $v(X') > |X'|$  and (using Schwarz inequality)  $\sum_{i=1}^n X'_i \frac{\partial v}{\partial X'_i} < |X'|$ , therefore  $w > 0$ . Moreover, up to lower order terms, we have:

$$\frac{\partial^2}{\partial X'_i \partial X'_j} (\delta v)(X') = w(X') \sum_{1 \leq i, j \leq n} \frac{\partial^2}{\partial x'_k \partial x'_l} (\delta\psi)(x') \frac{\partial x'_k}{\partial X'_i} \frac{\partial x'_l}{\partial X'_j}$$

with  $x'_k = \frac{X'_k}{\sqrt{v^2(X') - |X'|^2}}$ . We thus find in (4):  $B_{kl} = w(X') \sum_{1 \leq i, j \leq n} A_{ij} \frac{\partial x'_k}{\partial X'_i} \frac{\partial x'_l}{\partial X'_j}$

and the ellipticity of  $\delta\psi \mapsto dF_m(\psi)(\delta\psi)$  follows.  $\square$

We need also a more specific (ellipticity) property of the operator  $F_m$ , namely:

**Lemma 2.** *For each couple  $(\varphi_0, \varphi_1)$  of  $m$ -admissible real functions on  $\mathbb{H}$  and each point  $x_0 \in \mathbb{H}$  where  $\varphi = \varphi_1 - \varphi_0$  assumes a local extremum, the whole segment  $t \in [0, 1] \rightarrow \varphi_t = \varphi_0 + t\varphi$  consists of  $m$ -admissible functions at the point  $x_0$ .*

*Proof :* The analogue of Lemma 2 is fairly standard in the Cartesian setting, using the expression of the operator  $G_m$  introduced in the proof of Lemma 1 (see [2]) together with the well-known fact:  $\forall \kappa \in \Gamma_m, \forall i \in \{1, \dots, n\}, \frac{\partial \sigma_m}{\partial \kappa_i}(\kappa) > 0$ . Here, we will simply reduce the proof to that setting (and let the reader complete the argument). Let us first normalize the situation at an extremum point  $x_0 \in \mathbb{H}$  of  $\varphi$ . From (3), we may assume  $\varphi(x_0) = 0$ . Moreover, we may assume that  $\varphi$  has a local minimum at  $x_0$  (if not, switch  $\varphi_0$  and  $\varphi_1$ ). Finally, setting  $\text{graph}_{\mathbb{H}} \varphi_a = \text{graph}_{\mathbb{R}^n} u_a$  for  $a = 0, 1$ , and performing if necessary a suitable Lorentz transform (hyperbolic rotation), we may take  $x_0 = (0, 1) \in \mathbb{R}^n \times \mathbb{R}$  thus with  $u_a(0) = 1$ . For  $t \in [0, 1]$  and near  $x_0$ , set  $\Sigma_t = \text{graph}_{\mathbb{R}^n} u_t$  for the hypersurface  $\text{graph}_{\mathbb{H}} \varphi_t$ . We must prove that

$\Sigma_t$  is  $m$ -admissible at  $x_0$ . For  $X_t \in \mathbb{R}^{n,1}$  lying in  $\Sigma_t$ , we have:  $\overrightarrow{OX_t} = e^{t\varphi(x)} \overrightarrow{OX_0}$  with  $\overrightarrow{Ox} = \frac{\overrightarrow{OX_0}}{\sqrt{-|\overrightarrow{OX_0}|^2}}$ . In the Cartesian setting, we thus have (sticking to the  $\mathbb{R}^n$ -valued charts used in the preceding proof):

$$u_t(X'_t) = e^{t\varphi(x')} u_0[e^{-t\varphi(x')} X'_t],$$

here with  $x' = \frac{X'_0}{\sqrt{u_0^2(X'_0) - |X'_0|^2}}$ ,  $X'_t = e^{t\varphi(x')} X'_0$ , and  $(X'_0, u_0(X'_0)) \in \text{graph}_{\mathbb{R}^n} u_0$ ; moreover, the lemma boils down to proving that  $u_t$  is  $m$ -admissible at  $X'_t = 0$ . A routine calculation yields at  $X'_t = 0$  the equalities:

$$\frac{\partial u_t}{\partial X'_{ti}}(0) = \frac{\partial u_0}{\partial X'_{0i}}(0), \quad \frac{\partial^2 u_t}{\partial X'_{ti} \partial X'_{tj}}(0) = \frac{\partial^2 u_0}{\partial X'_{0i} \partial X'_{0j}}(0) + t \frac{\partial^2 \varphi}{\partial x'_i \partial x'_j}(0),$$

where, in the second one, the matrix  $\left[ \frac{\partial^2 \varphi}{\partial x'_i \partial x'_j}(0) \right]_{1 \leq i, j \leq n}$  is non-negative. The rest of the proof is now standard, thus omitted.  $\square$

**Theorem 2.** *The operator  $F_m$  is one-to-one on  $m$ -admissible functions of class  $C^2$  vanishing at infinity.*

*Proof :* Let us argue by contradiction. Let  $\varphi_0, \varphi_1$  be two  $m$ -admissible  $C^2$  functions vanishing at infinity and having the same image by  $F_m$ . For  $t \in [0, 1]$ , set  $\varphi_t = \varphi_0 + t\varphi$  with  $\varphi = \varphi_1 - \varphi_0$ . Since  $\varphi$  vanishes at infinity, if  $\varphi \not\equiv 0$ , it assumes a nonzero local extremum (a maximum, say, with no loss of generality) at some point  $x_0 \in \mathbb{H}$ . By Lemma 2, the whole segment  $t \in [0, 1] \rightarrow \varphi_t$  is  $m$ -admissible in a neighborhood  $\Omega$  of  $x_0$  where  $\varphi$  thus satisfies the second order linear equation  $L\varphi = 0$  with  $L1 = 0$  and the operator  $L$  given by  $L = \int_0^1 dF_m(\varphi_t) dt$ . Combining Lemma 1 above with Hopf's strong Maximum Principle (see [7]), we get  $\varphi \equiv \varphi(x_0)$  throughout  $\Omega$ . By connectedness, we infer  $\varphi \equiv \varphi(x_0) \neq 0$  on the whole of  $\mathbb{H}$ , contradicting  $\lim_{r(x) \rightarrow +\infty} \varphi = 0$ . So, indeed, we must have  $\varphi \equiv 0$ , in other words  $F_m$  is one-to-one.  $\square$

## 2. EXISTENCE OF A SOLUTION REDUCED TO THAT OF UPPER AND LOWER SOLUTIONS

**Theorem 3.** *Let  $h : \mathbb{H} \rightarrow \mathbb{R}$  be a function of class  $C^{2,\alpha}$ , for some  $\alpha \in (0, 1)$ , such that there exists  $\varphi^- \in C^{4,\alpha}(\mathbb{H})$  with  $\text{graph}_{\mathbb{H}} \varphi^-$  strictly convex and spacelike, and  $\varphi^+ \in C^2(\mathbb{H})$  with  $\text{graph}_{\mathbb{H}} \varphi^+$  spacelike, satisfying*

$$F_2(\varphi^-) \geq h, \quad F_2(\varphi^+) \leq h \quad \text{and} \quad \lim_{r(x) \rightarrow +\infty} \varphi^\pm = 0.$$

*Then the equation*

$$F_2(\varphi) = h$$

*has a unique admissible solution of class  $C^{4,\alpha}$  such that  $\lim_{r(x) \rightarrow +\infty} \varphi(x) = 0$ . Moreover  $\varphi$  satisfies the pinching:*

$$\varphi^- \leq \varphi \leq \varphi^+.$$

**Remark 2.** Since  $\varphi$  is a bounded function, the hypersurface  $M = \text{graph}_{\mathbb{H}}(\varphi)$  is entire. More precisely, denoting by  $\varphi_{\min}$  and  $\varphi_{\max}$  two constants such that  $\varphi_{\min} \leq \varphi \leq \varphi_{\max}$ , the function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\text{graph}_{\mathbb{R}^n}(u) = \text{graph}_{\mathbb{H}}(\varphi)$  satisfies  $u_{\min} \leq u \leq u_{\max}$  where  $u_{\min}$  (resp.  $u_{\max}$ ) is such that  $\text{graph}_{\mathbb{R}^n}(u_{\min}) = \text{graph}_{\mathbb{H}}(\varphi_{\min})$  (resp.  $\text{graph}_{\mathbb{R}^n}(u_{\max}) = \text{graph}_{\mathbb{H}}(\varphi_{\max})$ ). Noting that the graphs of

$u_{min}$  and  $u_{max}$  are hyperboloids, we see that the inequality  $u \geq u_{min}$  implies that  $M$  is entire, and the inequality  $u \leq u_{max}$  implies that  $M$  is asymptotic to the lightcone.

*Proof :* The asserted uniqueness follows from Theorem 2; so let us focus on the existence part. A straightforward comparison principle, using (4) and Lemma 2, implies  $\varphi^- \leq \varphi^+$  on  $\mathbb{H}$ . Let  $u^-, u^+ : \mathbb{R}^n \rightarrow \mathbb{R}$  be such that  $\text{graph}_{\mathbb{R}^n}(u^\pm) = \text{graph}_{\mathbb{H}}(\varphi^\pm)$ . Set  $H$  for the function on  $\mathbb{R}^{n,1}$  defined by:

$$(6) \quad H(X) = \frac{\binom{n}{2}}{|X_{n+1}|^2 - |X'|^2} \left[ h \left( \frac{X}{\sqrt{|X_{n+1}|^2 - |X'|^2}} \right) \right]^2.$$

The spacelike functions  $u^-$  and  $u^+$  satisfy:

$$H_2[u^-] \geq H(\cdot, u^-), \quad H_2[u^+] \leq H(\cdot, u^+), \quad u^- \leq u^+ \quad \text{and} \quad \lim_{|x'| \rightarrow \infty} [u^\pm(x') - |x'|] = 0,$$

where  $H_2[u^\pm]$  stands for the second mean curvature of the graph of  $u^\pm$ . Theorem 1.1 in [3] asserts the existence of a function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$ , belonging to  $C^{4,\alpha}$ , space-like, such that  $H_2[u] = H(\cdot, u)$  in  $\mathbb{R}^n$ ,  $\lim_{|x'| \rightarrow +\infty} u(x') - |x'| = 0$ , and  $u^- \leq u \leq u^+$ .

The function  $\varphi : \mathbb{H} \rightarrow \mathbb{R}$  such that  $\text{graph}_{\mathbb{H}}(\varphi) = \text{graph}_{\mathbb{R}^n}(u)$  is a solution of our original problem.  $\square$

### 3. CONSTRUCTION OF RADIAL UPPER AND LOWER SOLUTIONS

In the sequel of the paper, we first solve the Dirichlet problem on a bounded set in  $\mathbb{H}$  (section 3.1) then proceed to proving the existence and uniqueness of an entire solution in the radial case and study its properties (sections 3.2 and 3.3); finally, we construct the required radial barriers (section 3.4).

#### 3.1. The Dirichlet problem.

**Theorem 4.** *Given  $\alpha \in (0, 1)$ , let  $\Omega$  be a uniformly convex bounded open subset of  $\mathbb{H}$  with  $C^{2,\alpha}$  boundary,  $h : \Omega \rightarrow \mathbb{R}$  be a positive function of class  $C^{2,\alpha}$ , and  $\varphi_0 : \overline{\Omega} \rightarrow \mathbb{R}$  be a spacelike function of class  $C^{2,\alpha}$  whose radial graph is strictly convex. Then the Dirichlet problem*

$$(7) \quad F_2(\varphi) = h \text{ in } \Omega, \quad \varphi = \varphi_0 \text{ on } \partial\Omega,$$

*has a unique admissible solution of class  $C^{4,\alpha}$ .*

*Proof :* The proof of uniqueness follows the lines of the proof of Theorem 2; let us focus on the existence part. Setting  $x = (x', \sqrt{1 + |x'|^2}) \in \mathbb{R}^n \times \mathbb{R}$ , and

$$\Omega' = \{e^{\varphi_0(x)} x', \quad x \in \Omega\}, \quad u_0(e^{\varphi_0(x)} x') = e^{\varphi_0(x)} \sqrt{1 + |x'|^2},$$

problem (7) is equivalent to the Dirichlet problem:

$$(8) \quad H_2[u] = H(\cdot, u) \text{ in } \Omega', \quad u = u_0 \text{ on } \partial\Omega',$$

where  $H_2$  is the scalar curvature operator acting on spacelike graphs defined on  $\Omega' \subset \mathbb{R}^n$ , and  $H$  is defined on  $\Omega' \times \mathbb{R}$  by (6). We know essentially from [2, 14] that this problem is solvable (with an adaptation here because the function  $H$  depends also on  $u$ ; the existence is proved by a classical fixed point argument [7] and the required *a priori* estimates are carried out in [3, p.251]).  $\square$

**3.2. Existence and uniqueness of entire radial solutions.** The aim of this section is to prove the following result :

**Theorem 5.** *For  $\alpha \in (0, 1)$ , let  $h : \mathbb{R}^+ \rightarrow \mathbb{R}$  be a positive function of class  $C^{2,\alpha}$  constant on some neighborhood of 0 and let  $\varphi_0$  be a real number. Recall  $r = r(x)$  denotes the hyperbolic distance of  $x \in \mathbb{H}$  from a fixed origin  $o \in \mathbb{H}$ . The problem:*

$$(9) \quad F_2(\varphi)(x) = h(r) \text{ for all } x \in \mathbb{H}, \quad \varphi(o) = \varphi_0,$$

*admits a unique admissible radial solution  $\varphi : \mathbb{H} \rightarrow \mathbb{R}$  of class  $C^{4,\alpha}$ .*

*Proof : Existence:* let  $B_i$  denote the ball in  $\mathbb{H}$  with center  $o$  and radius  $i \in \mathbb{N}^*$ , and  $\varphi_i$  be the admissible solution of the Dirichlet problem:

$$(10) \quad F_2(\varphi) = h, \quad \varphi|_{\partial B_i} = 0,$$

given by Theorem 4. By radial symmetry and uniqueness,  $\varphi_i$  is a radial function:  $\varphi_i(x) = f_i(r)$  for some function  $f_i : [0, i] \rightarrow \mathbb{R}$ . By uniqueness again, for  $j > i$ , the function  $\varphi_j - \varphi_i$  must be constant on  $B_i$ . Therefore  $f_j'(r) \equiv f_i'(r)$  for  $r \in [0, i]$ . We may thus define  $g$  on  $\mathbb{R}^+$  by  $g = f_i'$  on each  $[0, i]$ . Now the function  $\varphi$  defined by

$$\varphi(x) = \varphi_0 + \int_0^r g(u) du$$

is a radial solution of (9).

*Uniqueness:* assume that  $\varphi_1$  and  $\varphi_2$  are admissible radial solutions of (9):  $\varphi_1(x) = f_1(r)$ ,  $\varphi_2(x) = f_2(r)$  where  $f_1, f_2$  are functions  $\mathbb{R}^+ \rightarrow \mathbb{R}$ . For each real  $R > 0$ , set

$$\varphi_{1,R}(x) = - \int_r^R f_1'(u) du \quad \text{and} \quad \varphi_{2,R}(x) = - \int_r^R f_2'(u) du.$$

The functions  $\varphi_{1,R}$  and  $\varphi_{2,R}$  are both admissible solutions of the Dirichlet problem (10) on  $B_R$ . As such, they must coincide on  $B_R$ , hence  $f_1' = f_2'$  on  $[0, R]$ , which implies the desired result.  $\square$

**3.3. Properties of the radial solutions.** The following lemma describes the monotonicity of a solution  $\varphi$  of equation (9) depending on the sign of  $h - 1$  :

**Lemma 3.** *Let  $h : \mathbb{R}^+ \rightarrow \mathbb{R}$  and  $\varphi : \mathbb{H} \rightarrow \mathbb{R}$  be as in Theorem 5, and let  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  be such that  $\varphi(x) = f[r(x)]$ ,  $\forall x \in \mathbb{H}$ .*

- (i) *If  $h \leq 1$ , then  $f$  is non-increasing; in particular, if  $\varphi_0 = 0$ , the function  $\varphi$  is non-positive.*
- (ii) *If  $h \geq 1$ , then  $f$  is non-decreasing; in particular, if  $\varphi_0 = 0$ , the function  $\varphi$  is non-negative.*

*Proof :* Here, we need to calculate explicitly the expression of equation (9) in the radial case. Fix  $x \in \mathbb{H}$  and take, with no loss of generality,

$$o = e_{n+1} = (0, \dots, 0, 1), \quad x = (\sinh r, 0, \dots, 0, \cosh r)$$

with  $r$ , the hyperbolic distance between  $o$  and  $x$ . Consider the orthonormal basis of  $T_x \mathbb{H}$  defined by:

$$\partial_r = \cosh r \, e_1 + \sinh r \, e_{n+1}, \quad \text{and} \quad \partial_\vartheta = e_\vartheta, \quad \vartheta = 2, \dots, n,$$

and the vectors, tangent to  $M = \text{graph}_{\mathbb{H}} \varphi$  at  $e^{\varphi(x)} x$ , induced by the embedding  $x \in \mathbb{H} \rightarrow e^{\varphi(x)} x \in M$ , given by:

$$u_r = e^f (f' x + \partial_r), \quad u_\vartheta = e^f \partial_\vartheta, \quad \vartheta = 2, \dots, n.$$

The future oriented unit normal to  $M$  at  $e^{\varphi(x)}x$  is the vector:

$$(11) \quad N(r) = \frac{f'}{\sqrt{1-f'^2}} \partial_r + \frac{1}{\sqrt{1-f'^2}} x .$$

Let  $S$  be the curvature endomorphism of  $M$  at  $e^{\varphi(x)}x$ , with respect to the future unit normal  $N(r)$ . Using the formulas

$$D_{\partial_r} \partial_r = x, \quad D_{\partial_\vartheta} \partial_r = \frac{1}{\tanh r} \partial_\vartheta$$

where  $D$  denotes the canonical flat connection of  $\mathbb{R}^{n,1}$ , we readily get:

$$S(u_r) = dN(\partial_r) = \frac{e^{-f}}{\sqrt{1-f'^2}} \left( \frac{f''}{1-f'^2} + 1 \right) u_r ,$$

and, for  $\vartheta = 2, \dots, n$ ,

$$S(u_\vartheta) = dN(\partial_\vartheta) = \frac{e^{-f}}{\sqrt{1-f'^2}} \left( \frac{f'}{\tanh r} + 1 \right) u_\vartheta .$$

The principal curvatures of  $M$  at  $r > 0$  are thus equal to:

$$\frac{e^{-f}}{\sqrt{1-f'^2}} \left( \frac{f''}{1-f'^2} + 1 \right) \text{ (simple)}, \quad \frac{e^{-f}}{\sqrt{1-f'^2}} \left( \frac{f'}{\tanh r} + 1 \right) \text{ (multiplicity } n-1 \text{)}.$$

Setting  $s = s(r)$  for the hyperbolic distance from  $o$  to  $N(r)$ , we infer from (11):

$$(12) \quad s(r) = r + \operatorname{Argth}(f').$$

In terms of the new radial unknown  $s(r)$ , for  $r > 0$ , the principal curvatures reads

$$(13) \quad \left( e^{-f} \cosh(r-s) s', e^{-f} \frac{\sinh s}{\sinh r}, \dots, e^{-f} \frac{\sinh s}{\sinh r} \right) ,$$

and the equation  $F_2(\varphi) = h$  reads

$$(14) \quad 2s' \cosh(r-s) \sinh r \sinh s = nh^2 \sinh^2 r - (n-2) \sinh^2 s.$$

We now prove the first statement of the lemma. Since  $f' = \tanh(s-r)$ , we must prove:  $s \leq r$  on  $[0, +\infty)$ . Suppose first  $h < 1$ . Since  $s(0) = 0$  and  $s'(0) = h(0) < 1$  (from (14)), there exists  $r_0 > 0$  such that  $s \leq r$  on  $[0, r_0]$ . Moreover, we get from (14):

$$s' \leq \frac{1}{2 \cosh(r-s)} \left( n \frac{\sinh r}{\sinh s} - (n-2) \frac{\sinh s}{\sinh r} \right) .$$

We observe that the function  $s(r) = r$  is a solution of the ODE:

$$s' = \frac{1}{2 \cosh(r-s)} \left( n \frac{\sinh r}{\sinh s} - (n-2) \frac{\sinh s}{\sinh r} \right)$$

on  $[r_0, +\infty)$ . So the comparison theorem for solutions of ordinary differential equations implies  $s \leq r$  on  $[r_0, +\infty)$ . Suppose only  $h \leq 1$ , fix  $A > 0$  and consider  $h_\delta = h - \delta$ , where  $\delta$  is some small positive constant such that  $h_\delta > 0$  on  $[0, A]$ . Denoting by  $\varphi_\delta$  and  $s_\delta$  the corresponding solutions of (9) and (14) on the ball of radius  $A$ , the function  $s_\delta - r$  is non-positive; we now prove that  $s_\delta - r$  converges uniformly to  $s - r$  as  $\delta$  tends to zero, which will yield the desired result. Set  $\overline{B}_A$  for the ball of radius  $A$  in  $\mathbb{H}$  and  $U = \{\psi \in C^{2,\alpha}(\overline{B}_A), \psi|_{\partial B_A} = 0\}$ ; consider the auxiliary map:

$$\Phi : \psi \in U \rightarrow \Phi(\psi) := F_2(\psi + \varphi) \in C^\alpha(\overline{B}_A) .$$

Since  $\Phi(0) = h$  and since, classically [7] (recalling (4)), the linearized map  $d\Phi(0)$  is an isomorphism from  $\{\xi \in C^{2,\alpha}(\overline{B}_A), \xi|_{\partial B_A} = 0\}$  to  $C^\alpha(\overline{B}_A)$ , the inverse function theorem implies:  $\forall \varepsilon > 0, \exists \delta_0 > 0, \forall \delta \in (0, \delta_0)$ , the solution  $\psi_\delta \in U$  of  $F_2(\psi_\delta + \varphi) = h_\delta$  satisfies  $|\psi_\delta|_{2,\alpha} \leq \varepsilon$ . Since  $\varphi_\delta = \psi_\delta + \varphi - \psi_\delta(o)$ , we obtain



$|\varphi_\delta - \varphi|_{2,\alpha} \leq 2\varepsilon$ , which implies the convergence of  $\varphi_\delta$  to  $\varphi$  in  $C^1$  and thus the uniform convergence of  $s_\delta$  to  $s$ .

The proof of statement (ii) is analogous and thus omitted  $\square$

Our next lemma provides a simple necessary and sufficient condition for an entire radial solution to be bounded.

**Lemma 4.** *Let  $h : \mathbb{R}^+ \rightarrow \mathbb{R}$  and  $\varphi : \mathbb{H} \rightarrow \mathbb{R}$  be as in Theorem 5.*

(i) *Assume  $h \leq 1$ , and  $\lim_{r \rightarrow \infty} h = 1$ . Then*

$$\lim_{r(x) \rightarrow +\infty} \varphi(x) > -\infty \text{ if and only if } \int_0^{+\infty} (1-h)dr \text{ converges.}$$

(ii) *Assume  $h \geq 1$ , and  $\lim_{r \rightarrow \infty} h = 1$ . Then*

$$\lim_{r(x) \rightarrow +\infty} \varphi(x) < +\infty \text{ if and only if } \int_0^{+\infty} (h-1)dr \text{ converges.}$$

*Proof :* Let us prove statement (i), thus assuming  $h \leq 1$ , with  $\lim_{r \rightarrow \infty} h = 1$ . We stick to the notations used in the proof of Lemma 3. From (12), we get at once:

$$(15) \quad \varphi(x) = \varphi_0 - \int_0^{r(x)} \tanh(u - s(u))du .$$

Statement (i) amounts to prove that  $\int_0^{+\infty} \tanh(u - s(u))du$  converges if and only if so does  $\int_0^{+\infty} (1-h)dr$ . We split the proof of this fact into five steps.

*Step 1:* the solution  $s$  of (14) is an increasing function.

Let us consider in the  $(r, s)$  plane the curve  $C$  with equation:

$$nh^2 \sinh^2 r = (n-2) \sinh^2 s, \quad r, s \geq 0.$$

The slope of its tangent at  $(0, 0)$  is  $\sqrt{\frac{n}{n-2}}h(0)$ . Since the solution  $s$  satisfies  $s(0) = 0$  and  $s'(0) = h(0)$ , we infer that the graph of  $s$  stays under the curve  $C$  near 0. Noting that the following vector field, associated to the differential equation (14):

$$(r, s) \mapsto (2 \cosh(r-s) \sinh r \sinh s, nh^2 \sinh^2 r - (n-2) \sinh^2 s) ,$$

is horizontal on  $C$ , and that the height  $s$  of the curve  $C$  is increasing with  $r$ , we conclude that the solution  $s$  of (14) remains trapped below  $C$ . In other words  $nh^2 \sinh^2 r \geq (n-2) \sinh^2 s$  for all  $r$ , and (14) implies:  $s' \geq 0$ .

*Step 2:*  $r - s$  has a limit at  $+\infty$ .

By contradiction, assume  $\liminf(r-s) < \limsup(r-s) = \delta$ . Thus there exists a sequence  $r_k \rightarrow +\infty$  such that  $r_k - s(r_k) \rightarrow \delta$  and  $s'(r_k) = 1$ . Denoting  $s(r_k)$  by  $s_k$ , we get from equation (14):

$$(16) \quad 1 = \frac{1}{2 \cosh(r_k - s_k)} \left[ nh^2(r_k) \frac{\sinh r_k}{\sinh s_k} - (n-2) \frac{\sinh s_k}{\sinh r_k} \right] .$$

We distinguish two cases :

*First case:*  $\delta < +\infty$ . We then have  $s_k \rightarrow +\infty$ ,  $\frac{\sinh r_k}{\sinh s_k} \sim e^{r_k - s_k} \sim e^\delta$  and  $\frac{\sinh s_k}{\sinh r_k} \sim e^{s_k - r_k} \sim e^{-\delta}$  as  $k$  tends to infinity (here and below, the equivalence  $\sim$  between two quantities means that their quotient has limit 1). So (16) yields

$$1 = \frac{1}{2 \cosh \delta} [ne^\delta - (n-2)e^{-\delta}] .$$

Using  $e^\delta \geq e^{-\delta}$  we get  $1 \geq \frac{e^\delta}{\cosh \delta}$ , which is absurd.

*Second case*  $\delta = +\infty$ . First assuming that  $s_k$  is not bounded, and since  $s$  is an increasing function (Step 1), we have :  $s_k \rightarrow +\infty$ ,  $\frac{\sinh r_k}{\sinh s_k} \sim e^{r_k - s_k} \rightarrow +\infty$  and  $\frac{\sinh s_k}{\sinh r_k} \sim e^{s_k - r_k} \rightarrow 0$  as  $k$  tends to infinity. Equation (16) yields

$$1 \sim \frac{n}{2 \cosh(r_k - s_k)} e^{r_k - s_k},$$

which is absurd since  $\cosh(r_k - s_k) \sim \frac{e^{r_k - s_k}}{2}$ . If we now assume  $s_k$  bounded, since  $s$  is an increasing function with  $s'(0) > 0$ , we get that  $s_k$  converges to  $l > 0$ , and, since  $\frac{\sinh s_k}{\sinh r_k} \rightarrow 0$ , we obtain from (16):

$$1 \sim \frac{n}{2 \cosh(r_k - s_k)} \frac{\sinh r_k}{\sinh l},$$

with  $\sinh r_k \sim \frac{e^{r_k}}{2}$ ,  $\cosh(r_k - s_k) \sim \frac{e^{r_k - s_k}}{2} \sim \frac{e^{-l}}{2} e^{r_k}$ ; so  $1 = \frac{n}{2} \frac{e^l}{\sinh l}$ , which is absurd.

*Step 3:*  $r - s$  tends to 0 at infinity.

Having proved that  $r - s$  converges, let us set  $\delta = \lim_{r \rightarrow +\infty} r - s$  and prove by contradiction that  $\delta = 0$ . There are two cases :

*First case :*  $0 < \delta < +\infty$ . We get  $s \rightarrow +\infty$ , hence  $\frac{\sinh r}{\sinh s} \sim e^{r-s} \sim e^\delta$ ,  $\frac{\sinh s}{\sinh r} \sim e^{s-r} \sim e^{-\delta}$  as  $r$  tends to infinity, and thus, from (14):

$$s' \rightarrow \frac{1}{2 \cosh \delta} [ne^\delta - (n-2)e^{-\delta}].$$

The latter expression is larger than 1, which contradicts  $r \geq s$ .

*Second case :*  $\delta = +\infty$ . We first note that  $\frac{\sinh s}{\sinh r} \rightarrow 0$  (if  $s$  is bounded this is trivial; if  $s$  is not bounded,  $s \rightarrow +\infty$  since  $s$  is increasing, and we have  $\frac{\sinh s}{\sinh r} \sim e^{s-r} \rightarrow 0$  since  $r - s \rightarrow +\infty$ ). Moreover we have  $\liminf nh^2 \frac{\sinh r}{\sinh s} \geq n$  since  $r \geq s$ . We thus infer from equation (14):

$$s' \sim \frac{n}{2 \cosh(r-s)} \frac{\sinh r}{\sinh s}.$$

Assuming  $s \rightarrow +\infty$ , we get  $\frac{\sinh r}{\sinh s} \sim e^{r-s}$  and  $\cosh(r-s) \sim \frac{e^{r-s}}{2}$ , hence  $s' \sim n$ , which is impossible since  $s \leq r$ .

Finally, assuming  $s$  bounded yields  $s \rightarrow l > 0$ ; since  $r - s \rightarrow +\infty$ , we infer  $\cosh(r-s) \sim \frac{e^{r-s}}{2}$  and  $\sinh r \sim \frac{e^r}{2}$ , hence from (14),  $e^{-s}s' \sim \frac{n}{2} \frac{1}{\sinh l}$  and thus  $s' \sim \frac{n}{2} \frac{e^l}{\sinh l}$ , which contradicts the boundedness assumption on  $s$ .

*Step 4:*  $\lim_{r(x) \rightarrow +\infty} \varphi(x) > -\infty$  if and only if  $\varepsilon(r) := r - s$  is integrable on  $[0, +\infty)$ .

This is straightforward from (15) combined with  $\tanh(u - s(u)) \sim \varepsilon(u)$  which holds as  $u \rightarrow +\infty$  due to Step 3.

*Step 5:*  $\varepsilon$  is integrable on  $[0, +\infty)$  if and only if  $\beta := 1 - h^2$  is integrable on  $[0, +\infty)$ .

First observation:  $\lim_{r \rightarrow \infty} s' = 1$ . Indeed, at infinity, we have  $r - s \rightarrow 0$ , so  $s \rightarrow +\infty$ , hence:

$$\frac{\sinh r}{\sinh s} \sim e^{r-s} \sim 1, \quad \frac{\sinh s}{\sinh r} \sim e^{s-r} \sim 1,$$

and (14) yields  $s' \rightarrow 1$ .

Using Step 3, the assumptions on  $h$  and the preceding observation, we get

$$\varepsilon(r) \rightarrow 0, \beta(r) \rightarrow 0, \text{ and } \varepsilon'(r) = 1 - s'(r) \rightarrow 0$$

as  $r$  tends to infinity. Plugging the definitions of  $\varepsilon$  and  $\beta$  in (14) and using the expansions

$$\cosh \varepsilon = 1 + o(\varepsilon), \sinh(r - \varepsilon) = \sinh r (1 - \varepsilon + o(\varepsilon)),$$

yields

$$(17) \quad (n-1)\varepsilon + \varepsilon' + o(\varepsilon) = \frac{n}{2}\beta.$$

Fixing a real  $\delta > 0$ , there readily exists  $r_\delta > 0$  such that, for all  $r \geq r_\delta$ ,

$$(18) \quad \varepsilon' + (n-1-\delta)\varepsilon \leq \frac{n}{2}\beta,$$

and

$$(19) \quad \varepsilon' + (n-1+\delta)\varepsilon \geq \frac{n}{2}\beta.$$

Integrating (18), we get, for  $r \geq r_\delta$ ,

$$\varepsilon(r) \leq e^{-(n-1-\delta)r} \left[ C(r_\delta) + \frac{n}{2} \int_{r_\delta}^r \beta(u) e^{(n-1-\delta)u} du \right].$$

Integrating again and using Fubini Theorem yields, with  $\delta$  such that  $n-1-\delta > 0$ ,

$$\begin{aligned} \int_{r_\delta}^{+\infty} \varepsilon(r) dr &\leq C'(r_\delta) + \frac{n}{2} \int_{r_\delta}^{+\infty} \beta(u) e^{(n-1-\delta)u} \left( \int_u^{+\infty} e^{-(n-1-\delta)r} dr \right) du, \\ &\leq C'(r_\delta) + \frac{n}{2(n-1-\delta)} \int_{r_\delta}^{+\infty} \beta(u) du. \end{aligned}$$

We conclude that  $\varepsilon$  is integrable provided  $\beta = 1 - h^2$  is integrable.

Analogously, using (19), we get

$$\varepsilon(r) \geq e^{-(n-1+\delta)r} \left[ C(r_\delta) + \frac{n}{2} \int_{r_\delta}^r \beta(u) e^{(n-1+\delta)u} du \right],$$

and

$$\begin{aligned} \int_{r_\delta}^{+\infty} \varepsilon(r) dr &\geq C'(r_\delta) + \frac{n}{2} \int_{r_\delta}^{+\infty} \beta(u) e^{(n-1+\delta)u} \left( \int_u^{+\infty} e^{-(n-1+\delta)r} dr \right) du, \\ &\geq C'(r_\delta) + \frac{n}{2(n-1+\delta)} \int_{r_\delta}^{+\infty} \beta(u) du. \end{aligned}$$

Taking  $\delta > 0$  arbitrary, we find that  $\beta$  is integrable if  $\varepsilon$  is integrable.

The proof of statement (ii) is analogous and thus omitted  $\square$

### 3.4. Construction of appropriate radial barriers.

**Lemma 5.** *Let  $h : \mathbb{H} \rightarrow \mathbb{R}$  be a positive and continuous function on the hyperbolic space such that*

$$\lim_{r(x) \rightarrow +\infty} h(x) = 1$$

*and such that the functions  $h^-$  and  $h^+$  defined on  $\mathbb{R}^+$  by*

$$h^-(r) = \sup_{r(x)=r} h(x) \text{ and } h^+(r) = \inf_{r(x)=r} h(x)$$

*satisfy*

$$\int_0^{+\infty} (h^- - 1)_+ dr < +\infty, \int_0^{+\infty} (1 - h^+)_+ dr < +\infty,$$

where  $(h^- - 1)_+$  (resp.  $(1 - h^+)_+$ ) means the positive part of  $h^- - 1$  (resp.  $1 - h^+$ ). Then there exist  $\varphi^-, \varphi^+ \in C^\infty(\mathbb{H})$ , with strictly convex spacelike graphs, satisfying:

$$F_2(\varphi^-) \geq h, F_2(\varphi^+) \leq h \text{ and } \lim_{r \rightarrow +\infty} \varphi^\pm = 0.$$

*Proof :* First, considering  $1 + (h^- - 1)_+$  instead of  $h^-$  and  $1 - (1 - h^+)_+$  instead of  $h^+$ , we may suppose without loss of generality that  $h^-$  and  $h^+$  are two continuous functions such that :  $\forall x \in \mathbb{H}$ , with  $r = r(x)$ ,

$$(20) \quad h^-(r) \geq h(x) \geq h^+(r) > 0,$$

$$(21) \quad h^- \geq 1 \geq h^+, \quad \lim_{r \rightarrow +\infty} h^-(r) = \lim_{r \rightarrow +\infty} h^+(r) = 1,$$

and

$$(22) \quad \int_0^{+\infty} (h^- - 1)dr < +\infty, \quad \int_0^{+\infty} (1 - h^+)dr < +\infty.$$

If we now consider

$$h^- + \frac{\varepsilon_0}{r^2} \text{ if } r \geq 1, \quad h^- + \varepsilon_0 \text{ if } r \leq 1$$

instead of  $h^-$ , and

$$h^+ - \frac{\varepsilon_0}{r^2} \text{ if } r \geq 1, \quad h^+ - \varepsilon_0 \text{ if } r \leq 1$$

instead of  $h^+$ , where  $\varepsilon_0$  is chosen sufficiently small such that  $\inf h^+ > \varepsilon_0$ , we may moreover assume the following:

$$h^- \geq \max(1, h) + \frac{\varepsilon_0}{r^2} \text{ and } h^+ \leq \min(1, h) - \frac{\varepsilon_0}{r^2} \text{ if } r \geq 1.$$

We now prove that we can approximate  $h^\pm$  by smooth functions  $g^\pm$  such that

$$(23) \quad |h^\pm - g^\pm| \leq \min\left(\frac{\varepsilon_0}{r^2}, \varepsilon_0\right).$$

For each  $i \in \mathbb{N}$ , let us denote by  $g_i^-$  a smooth function on  $[0, i+1]$  such that  $|h^- - g_i^-| \leq \frac{\varepsilon_0}{(i+1)^2}$  on  $[0, i+1]$ . Let  $\vartheta \in C_c^\infty(\mathbb{R})$  such that  $0 \leq \vartheta \leq 1$ ,  $\vartheta(x) = 1$  if  $|x| \leq \frac{1}{4}$  and  $\vartheta(x) = 0$  if  $|x| \geq \frac{3}{4}$ . We define  $g^-$  on  $[i, i+1]$  by

$$g^- = \vartheta_i g_i^- + (1 - \vartheta_i) g_{i+1}^-,$$

where  $\vartheta_i = \vartheta(\cdot - i)$ . By construction, we have  $g^- = g_i^-$  on a neighborhood of  $i$ . The function  $g^-$  is thus smooth on  $[0, +\infty)$ , and satisfies on  $[i, i+1]$  :

$$|g^- - h^-| \leq \vartheta_i |g_i^- - h^-| + (1 - \vartheta_i) |g_{i+1}^- - h^-| \leq \frac{\varepsilon_0}{(i+1)^2},$$

which implies the estimate (23). We may thus assume that (20), (21) and (22) hold, where  $h^\pm$  are two smooth functions on  $\mathbb{R}^+$ . Considering  $\vartheta \sup_{\mathbb{R}^+} h^- + (1 - \vartheta)h^-$  instead of  $h^-$ , and  $\vartheta \inf_{\mathbb{R}^+} h^+ + (1 - \vartheta)h^+$  instead of  $h^+$ , we may also assume that the functions  $h^\pm$  are constant on some neighborhood of 0. Let  $\varphi^-$  and  $\varphi^+$  be smooth radial functions given by Theorem 5 (with some arbitrary initial condition  $\varphi_0$ ) such that  $F_2(\varphi^\pm) = h^\pm$ . From Lemma 4, subtracting constants if necessary, we obtain  $\lim_{r \rightarrow +\infty} \varphi^\pm(r) = 0$   $\square$

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